Topology and Geometry of Curves in Hyperbolic Surfaces

Christopher-Lloyd Simon

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Abstract

This set of notes introduces my research experience at Stony Brook university. I took part in the 2017's summer topology and geometry workshop animated by Moira Chas, Tony Phillips and Dennis Sullivan. I would like to thank them all for considering Etienne Ghys's recommendation, as well as for their supervising. I would also like to thank their students for sharing pleasurable mathematical moments.

My work focused on the relationships between the dynamics, the geometry and the topology of curves in hyperbolic surfaces. With Moira Chas, we proved a couple of experimentaly supported conjectures concerning the topology of curves and their Fricke polynomials in the pair of pants. In the process we constructed an algebra for multicurves on surfaces bearing striking relationships with quantum topology. It provides a general setting for understanding the relationship between Fricke polynomials, simple intersection numbers and the topology of curves; which we applied to the punctured torus. We thus wrote an article, to which this set of notes can serve as background and introduction.

Introduction

Free homotopy classes of closed curves in a topological 2-manifold of negative Euler characteristic share deep relationships with the space of hyperblic structures on that surface, and some of those are yet misunderstood.

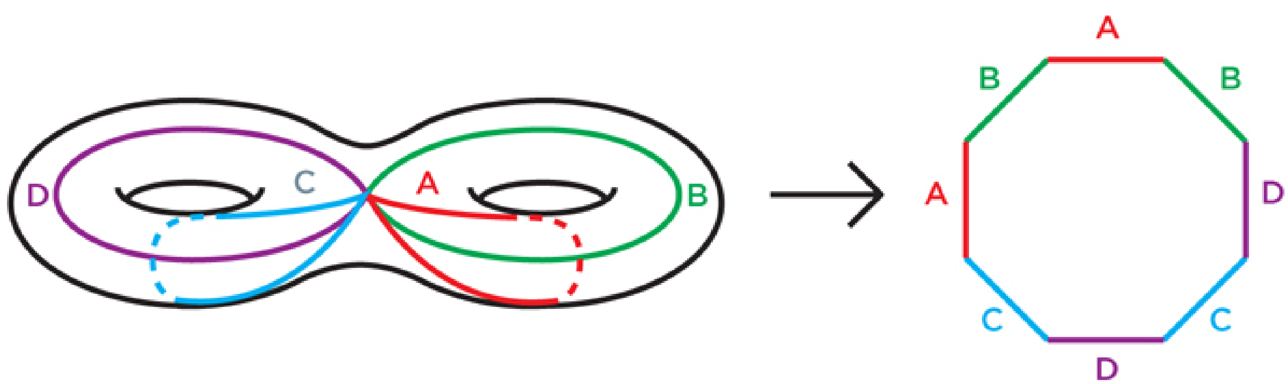
The choice of a hyperbolic metric enables us to study their dynamical properties with respect to the corresonding geodesic flow. The dynamics and the geometry are equivalent in the sense that we can compute lengths and areas knowing the dynamics of, and intersection numbers between, closed curves. The first part of this set of notes is dedicated to a formula i found, expressing the length of an arc as an expression involving the limit number of intersections with long closed geodesics. It involves ergodic properties of the geodesic flow so i shall first introduce the basics of ergodic theory.

Pushing in this direction, we can try to understand the combinatorics of intersection numbers between free homotopy classes of closed curves. This brought me to the more abstract setting of Teichmüller space, measured foliations and geodesic currents. The objective was to unravel some tight relationships between intersection numbers and the space of all hyperbolic metrics on a surface.

Here's couple of results we proved with Moira Chas [2] motivating the introduction of the hereafter defined notions. One can define two relations on the set of (homotopy classes of) closed curves: length equivalence and simple intersection equivalence. Two curves are length equivalent if they have the same length for any hyperbolic metric, and two curves are simple intersection equivalent if they have the same intersection number with any other simple curve. An algebraic procedure attaches to each curve a polynomial in several variables, called the Fricke polynomial, which characterises its length equivalence class. It is known that length equivalence implies simple intersection equivalence and we showed in elementary cases how to read a curve's simple intersection class off its Fricke polynomial. More generally we explained how to obtain simple intersection equivalence as a tropical limit of length equivalence and deduced, from an algebraic structure of multicurves in surfaces, a way to compute a tropical polynomial characterising a curves simple intersection class. We also found a state sum formula for the Fricke polynomial therefore giving a (quantum) topological understanding of length and simple intersection equivalence, as well as providing strong analogies with the Kauffman bracket in knot theory [3].

Preliminary notions

Throughout the paper we often work with a genus g surface S having$b$ boundary components or punctures, and negative Euler characteristic: $2g - 2 - b < 0$. Any metric considered on them should make geodesic boundaries. If $b > 0$ then its fundamental group is a free group on $2g + b - 1$ generators and if not it has presentation $< a_1, b_1, \dots, a_g, b_g \mid \prod [a_i, b_i] >$. One can start proving this for null genus and any number of boundary components, that is a punctured sphere, which is homotopically equivalent to a bouquet of circles. The case $b = 1$ is also easy to handle by cutting S into a $4g$-gon and taking away the the point in the middle as explained in [6]. Applying Van Kampen's theorem to appropriate connected sums gives all the other cases.



The free homotopy classes of closed curves on a surface will be denoted $\mathscr{C}$ and they correspond to the connected components of the space of continuous maps of a circle in the surface endowed with compact open topology. This space can also be identified with the conjugacy classes of the fundamental group π_1 (of the surface itself) and if this group is freely generated by Q then the elements of $\mathscr{C}$ are freely reduced cyclic words on Q. For any $w \in \mathscr{C}$ we can therefore define the word length in the generators Q as being the length of a shortest cyclic reduced word representing w. If the metric has negative curvature, each class in $\mathscr{C}$ contains a unique geodesic. Existence corresponds to finding a minimum of the energy functional in the class of w which follows from Ascoli's theorem, whereas unicity is due to negative curvature. The Milnor-Svarc's lemma gives a quasi-isometry between word length and geometric length for closed geodesics. The article [1] goes into more detail about these notions.

If $c : [0, 1] \to S$ is an arc in some surface, we denote $l(c)$ its length and $n_c(\gamma)$ its geometric intersection with another curve γ and if w is a homotopy class of curves we write $wl(w)$ its word length. Finally we denote by $\mathscr{S}$ the set of homotopy classes of simple curves in a surface.

Crofton-like formula in hyperbolic surfaces

Motivation

In the Euclidean plane, the Crofton formula (see [8] for much more on integral geometry) expresses the length of a rectifiable curve c as an integral over the first affine grassmanian, that is the space of lines, of the intersection number of c with each line. The space of lines is for that purpose endowed with the invariant Haar measure for the affine isometry group $d\gamma = \frac{1}{2} dr \wedge d\theta$ where $\gamma : \cos(\theta)\, x + \sin(\theta)\, y = r$.

$$l(c) = \int n_c(\gamma)\, d\gamma = \frac{1}{2} \iint n_c(r, \theta)\, dr\, d\theta$$

Santalo proved in [7] a similar formula in the hyperbolic plane. But in a closed hyperbolic surface, ergodicity of the geodesic flow for some well chosen invariant measure ensures that one (generic) geodesic is sufficient to cover the whole space in a uniform manner and one can therefore hope to find the length of c examinig the number of intersections with larger and larger portions of that one geodesic. From the Bowen-Margulis equidistribution property of closed geodesics we derive a summation formula over $w \in \mathscr{C}$ of the intersection numbers with c but it is essentially in the limit that w gets long that the information for calculating c 's length is encoded. Therefore, one can ask what additional information one can recover by considering the whole family of integers $(n_c(w))_{w \in \mathscr{C}}$ or by restricting this family to simple curves.

Ergodic dynamical systems

This section relies mostly on [9], defining and stating what we need to explain some Crofton-flavoured formulas in closed hyperbolic surfaces.

Measure preserving transformations and Poincaré's recurrence theorem Let $(M, \mathscr{T}, \mu)$ be a finitely measured space, and $T\colon\ M \to M$ a bijective measurable function. It is often the case that T is a homeomorphism on a compact Hausdorff space with Borel sigma algebra. The transformation T preserves μ if $T_* \mu = \mu$ which means that for every measurable set $A \in \mathscr{T}$ we have $\mu(T^{-1}A) = \mu(A)$. The measure preserving property can also be stated dualy in terms random variables by supposing that for any $f \in \mathscr{L}^1_\mu(M)$ we have $\int f\, d\mu = \int f \circ T\, d\mu$. Under these conditions Poincaré's recurrence lemma says that almost all points are recurrent. This means that if we consider any set $U \in \mathscr{T}$, then for almost all $x \in U$ there exists an infinite increasing sequence of integers (n_i) such that $T^{n_i}(x) \in U$. Note that this includes the case where $\mu(U) = 0$ in which case "for almost all" holds even if no points in U are recurrent.

Ergodic flows and Birkoff's theorem The right concept of indecomposability for a measure preserving dynamical system $(M, \mathscr{T}, \mu, T)$ is that T should be ergodic for μ : any $U \in \mathscr{T}$ satisfying $\mu(T^{-1}A) = \mu(A)$ must verify $\mu(A) \in \{0, 1\}$. The first main theorem is due to Birkoff and states that for such a system, the time average of the values of any function f along an T-orbit converges almost always to the space average of X . This is in fact equivalent to ergodicity.

Theorem. *If $(M, \mathscr{T}, \mu, T)$ is an ergodic dynamical system, then for any $f \in \mathscr{L}^1_\mu(M)$ and for almost all $x \in M$, the time averages $S_n(x) = \frac{1}{n}\sum_{0\leq k<n} f \circ T^k(x)$ converge to $\int f\, d\mu$.*

Mixing property The transformation T is mixing if for any $f,\, g \in \mathscr{L}^2_\mu(M)$, the functions $f \circ T^k$ and g become asymptotically independent for large k : $\int (f \circ T^k).g\, d\mu \to \left(\int f\, d\mu\right)\left(\int g\, d\mu\right)$. Applying this to the characteristic functions of measurable sets A and B means that the mass in A gets uniformily distributed in the measured space (M, μ). Mixing implies ergodicity implies measure preserving but none of these implications have reciprocals. If we define $d\nu_0 = \rho.d\mu$ (for a density function ρ) and consider it as the initial state of our system, then the mixing property sais that the future distribution states $\nu_k = (f \circ T^k)\, d\mu$ converge to a positive multiple of the invariant measure $d\mu$.

Continuous time flows Consider now a one parameter group $(\varphi^t)_{t\in\mathbb{R}}$ of measurable transformations acting on $(M, \mathscr{T}, \mu)$ such that if $f(x)$ is a measurable function on M then $f \circ \varphi^t(x)$ is a measurable function on $M \times \mathbb{R}$. We call $(\varphi^t)_{t\in\mathbb{R}}$ a flow. If the flow is measure preserving, that is if each φ^t preserves μ, then Poincaré's recurrence theorem still holds taking care to define a point $x \in U$ as recurrent if there exists an increasing sequence $t_i \to \infty$ such that $\varphi^{t_i}(x) \in U$. The flow is ergodic for μ if any measurable subset invariant under all the φ^t has null or full measure; and in that case Birkoff's theorem says that $S_t(x) = \frac{1}{t}\int_0^t f \circ \varphi^s(x)\, ds$ converges almost everywhere to $\int f\, d\mu$. The mixing property becomes for almost all $x \in M$, $\int (f \circ \varphi^t(x)).g\, d\mu \to \left(\int f\, d\mu\right)\left(\int g\, d\mu\right)$.

The geodesic flow

Ergodicity and Mixing Let's consider the unitary tangent bundle $M = T_1S$ of a surface $(S,\, h = ds^2)$. There is a natural distance function on M given locally for $x = (p, u);\; y = (q, v) \in M$ by $d(x, y)^2 = l(\gamma)^2 + h_q(tr_\gamma(u), v)^2$ where γ is the geodesic from p to q and $tr_\gamma(v)$ the parallel transport of the vector v along γ. This distance arises from a Riemannian metric which in turn gives a volume form $d\mu$ and a borelian measure μ on M named after Liouville. If S was a hyperbolic surface, this measure could also be constructed from the Haar measure of $Isom(\mathbb{H}^2) = PSL_2(\mathbb{R})$, since this group acts transitively and properly-discontinuously by isometries on S. The geodesic flow $\varphi^t(p, v)$ on (M, μ) preserves this Liouville measure.

Theorem. *(Hedlung-Hopf) The geodesic flow on a surface with (constant) negative curvature is ergodic and mixing for the measure μ.*

Equidistribution of closed geodesics Every closed geodesic γ defines a geodesic-flow-invariant measure defined by its action on functions:

$$\mu_\gamma(f) = \frac{1}{l(\gamma)}\int_\gamma f\, d\mu = \frac{1}{l(\gamma)}\int_0^{l(\gamma)} f(\varphi^t(x))\, dt$$

Let Γ_L be the set of closed geodesics of length smaller than L (finite set indexed by $\mathscr{C}$). Margulis showed that $card(\Gamma_L) \sim {}^{\exp(L)}/_{2L}$. The following result means that the closed geodesics uniformly fill in the space as they get longer.

Theorem. *(Bowen, Margulis) In constant negative curvature, the sequence of measures*

$$\Sigma_L = \frac{1}{card(\Gamma_L)} \sum_{\gamma \in \Gamma_L} \mu_\gamma$$

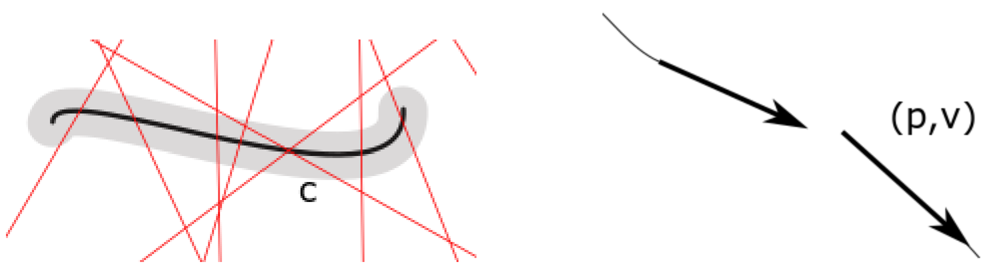
converges weakly to the Lebesgue measure σ on S (with volume form $d\sigma$).

Bowen-Margulis' theorem should be understood in a more general setting. The weighted sequence of normalised Dirac measures along closed geodesics always converges to a geodesic-flow invariant measure with maximal entropy. In negative curvature, this maximal entropy measure is unique and it corresponds with the Liouville measure if and only iff the curvature is constant.

Two formulas for curve length via geodesics

Main idea In a hyperbolic surface S, ergodicity of the geodesic flow gives a mean to compute the area of a region by considering the proportion of time that a generic geodesic passes through it. If the region looks like a thin rectangular strip around an arc, then entering the region or cutting the arc is about the same and the area of the region divided by its width is close to the arc's length. So the length of an arc is approximated by the average number intersections with a generic geodesic. Poicaré's recurrence theorem implies that closed geodesics can approximate arbitrarily well any geodesic, so that the limit number of intersections can be taken on a sequence of closed geodesics. Those ideas are illustrated in the following pictures.

Let $c : [0,1] \to S$ be any smooth curve and consider $V^\eta = V^\eta(c)$ the $(\eta/2)$-neighborhood of c in S. Since c is compact, $\sigma(V^\eta)/\eta \to l(c)$.



Limit formula Applying Hedlung-Hopf's theorem to the characteristic function χ^η of V^η and dividing by η, we get for almost any geodesic $\gamma(t) = \varphi^t(p,v)$:

$$\frac{1}{t} \int_0^t \frac{\chi^\eta(\gamma(u))}{\eta} \, du \xrightarrow[t]{} \frac{\sigma(V^\eta)}{\eta} \tag{1}$$

Fix some small positive ϵ, then choose η satisfying $|\sigma(V^\eta)/\eta - l(c)| < \epsilon$ and such that for all large enough t the left hand side in 1 differs from $\frac{1}{t} n_c(\gamma_{[0,t]})$ of at most ϵ. I assume this last condition is possible to realize since γ can be chosen to verify both the mixing property and to intersect c transversally with an angle statistically close to $\pi/2$. This angle distribution is concentrated so that long incursions in V^η will be scarse enough to average out over time (see [5]).

Chose t such that 1 is an equality $\pm\epsilon$: triangle inequality gives$\left|\frac{1}{t} n_c(\gamma_{[0,t]}) - l(c)\right| < 3\epsilon$ so that $\frac{1}{t} n_c(\gamma_{[0,t]}) \to l(c)$. By the Poincaré recurrence theorem we can chose a strictly increasing sequence to infinity (t_n) such that $(\gamma(t), \dot{\gamma}(t))$ passes very close to its initial value(p, v). By connecting $\gamma(t_n)$ to p and taking the closed geodesic w_n in the same homotopy class we get a curve which stays close to $\gamma_{[0,t_n]}$ for large t_n since this last is alsmost a shortest path. So chosing p far from c we have for large n: $n_c(w_n) = n_c(\gamma_{[0,t_n]})$. Hence ${n_c(w_n)}/{l(w_n)} \to l(c)$.

Summation formula It is tempting to generalise the previous result as a limit on any word $w \in \mathscr{C}$ and say $\lim\limits_{w\in\mathscr{C}} {n_c(w)}/{l(w)} = l(c)$. This is not true since one can consider a sequence of very long simple curves which won't fill in the whole space. But we know that on average they do fill the space in a uniform way, so we can hope for a similar formula averaging on geodesics.

Applying Bowen-Margulis' theorem to the characteristic function f of V^ϵ we get:

$$\frac{1}{card(\Gamma_L)} \sum_{\gamma\in\Gamma_L} \chi^\eta(\gamma) \to \sigma(V^\eta)$$

Dividing by η and inverting the limits $\eta \to 0$ and $L \to \infty$ as before we find:

$$\frac{1}{card(\Gamma_L)} \sum_{\gamma\in\Gamma_L} n_c(\gamma) \to l(c)$$

Here's another (yet incomplete) attempt to prove this, avoiding the knowledge of the angle distribution at intersections which serves the inversion of limits, but using notions about geodesic currents introduced later on. Let's work in the universal cover $\tilde{S} = \mathbb{H}$ and use Otal's "Crofton formula" ([4], proposition 3). This formula asserts that the length of a closed geodesic is equal to the intersection form between the geodesic current it induces and the Liouville current associated to the metric on the surface. Liouville's current is constructed by lifting the metric m to the unit tangent bundle of the universal cover, and contracting its associated volume form with the geodesic vector field in order to get a transverse measure to the geodesic flow, hence a current λ_m. One can imagine measuring the length of a yellow fin in Esher's circle limit III picture by estimating the proportion of circles orthogonal to the boundary which intersect one chosen fin somewhere in the drawing.

Consider the case where c is a geodesic arc in S and let $\tilde{c}$ be one of its lifts to the hyperbolic plane. The length of c is thus equal to the Liouville measure of the set of geodesics of $\tilde{S}$ intersecting $\tilde{c}$, formally: $\lambda_m\left(\{g \in G(\tilde{S}); g \cap \tilde{c} \neq \emptyset\}\right)$. The sum Σ_L can be seen as a weighted multicurve, and therefore also as a geodesic current. However, although the theorem ensures that Σ_L converges weakly to the Lebesgue measure σ_m , what we need now is a weak convergence towards λ_m in the space of currents. Assuming this is true would precisely give the desired formula and the case where c is a rectifiable arc follows by approximation.

Finally, i think one could also prove the summation result by using density (by Poincaré's lemma) and some correct notion of equidistribution with respect to the metric for the lifts of all closed geodesics inside the set of all geodesics of the hyperbolic plane (by Bowen-Margulis). Then the integral in Santalo's hyperbolic Crofton formula could then be approximated by the summation expression obtained from Bowen-Margulis' theorem.

Mapping class group and Teichmüller space

Structures on closed surfaces

A closed oriented surface S (compact oriented topological 2-manifold) is characterised, modulo homeomorphism, by its genus.

Conformal structures Such a topological manifold can always be endowed with a Riemannian metric by averaging with a partition of unity some chart pull-backs of a metric in the plane. Riemannian metrics provide a measurement of angles in the tangent spaces, and two metrics yield the same angle-measurement iff they are conformally equivalent, that is they differ by multiplication of a positive function. Choosing a measurement of angles on the tangent bundle is called a conformal structure, so conformal structures are the same as Riemannian metrics up to multiplication by positive functions.

Complex structures It is remarkable that any closed orientable surface can be realized as a Riemann surface: one can chose a new compatible atlas of charts with values in the complex plane whose transition functions are holomorphic. Compatible means that the transition maps arising from an old chart and a new chart are homeomorphisms (or diffeomorphisms if we start with a smooth manifold). Such an atlas is called a complex structure on S and it also defines a way of measuring angles in the tangent space at every point. This is because the transition maps are conformal (their differential preserve angles) so that one can pull-back consistently the angle-measurement from the complex plane to the surface. The conformal class of metrics hereby associated to a complex structure is such that the multiplications by i in the tangent planes are isometries (the metrics are Hermitean for the Riemann surface) so in some sense the infinitesimal circles defined by the conformal class of metrics and the complex structure are the same.

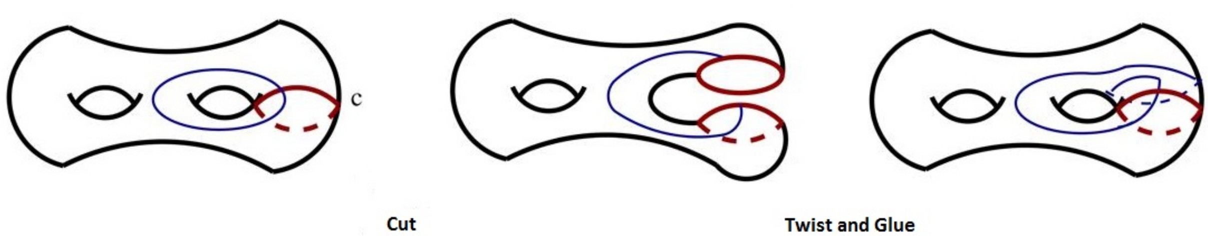
Hyperbolic structures Another remarkable fact which is a corolary of the uniformisation theorem, is that if the genus is greater than two, then any conformal class of metrics contains a unique metric of constant -1 curvature called a hyperbolic metric (because it induces a local isometry with the hyperbolic plane). So conformal structures up to conformal mappings, complex structures up to biholomorphic mappings and hyperbolic metrics on S considered up to isometry are all the same. This happens only in dimension 2 where the conformal group of the unit disc $Möb(\Delta)$, the automorphisms of the hyperbolic plane $Aut\,(\mathbb{H})$ and the isometries of Poincaré's metric $Isom\left(\Delta, \frac{2|dz|}{(1-|z|^2)}\right)$ all miraculously coïncide.

Given a hyperbolic metric on S, the developing map $Dev : \tilde{S} \to \mathbb{H}$ and the holonomy map $hol : \pi_1(S) \to Aut\,(\mathbb{H})$ identify its universal cover with a subset of the hyperbolic plane and its fundamental group with a subgroup of hyperbolic isometries (defined by $Dev \circ \alpha = hol(\alpha)$ for any $\alpha \in \pi_1(S)$). Since S is complete, it is isometric to $\Gamma\backslash\mathbb{H}$ for some Fuchsian group $\Gamma :$ its fundamental group. Reciprocally, a local homeomorphism $Dev : \tilde{S} \to \mathbb{H}$ and a homomorphism $hol : \pi_1(S) \to Aut\,(\mathbb{H})$ such that $Dev \circ \alpha = hol(\alpha)$ for any $\alpha \in \pi_1(S)$ defines a unique hyperbolic structure with those developping map and holonomy. A hyperbolic metric on S is therefore equivalent to a faithfull representation of its fundamental group into a discrete subgroup of $Aut(\mathbb{H})$, defined up to conjugation. From now on, S shall always be a closed topological surface with genus greater than two.

Spaces of structures

Teichmuller space We call (X, f) a marked pair if $f : S \to X = \Gamma \setminus \mathbb{H}$ is a homeomorphism to a hyperbolic surface. Marked pairs are tantamount to marked hyperbolic structures on S by pulling back the metric on X with the marking homeomorphism f , and the holonomy map is then given by $f_* : \pi_1(S) \to \Gamma$. We define Teichmuller space $\mathcal{T}(S)$ as the set of all marked pairs up to the equivalence relation $(X, f) \sim (Y, g)$ if $g \circ f^{-1}$ is isotopic to an isometry between X and Y. Put differently, it is the set of marked hyperbolic metrics up to composition by isometries isotopic to the identity. Since equivalent marked pairs give rise to conjugate holonomy maps, Teichmuller space injects in $Hom\,(\pi_1(S), Aut\,(\mathbb{H}))/\,conjugation$. So we can successively put the compact open topology on $Aut\,(\mathbb{H})$ (homeomorphic to the natural one on $PSL_2(\mathbb{R})$) and the discrete one on $\pi_1(S)$, then the compact open topology on $Hom\,(\pi_1(S), Aut\,(\mathbb{H}))$ and finally the quotient topology on the resulting space to define a natural topology on $\mathcal{T}(S)$. The topology thus defined is separated and we shall later on construct some compatible coordinates on $\mathcal{T}(S)$.

Mapping class group Define the mapping class group $MCG\,(S) = Homeo^+(S)/\,Homeo_0(S)$ to be the group of orientation preserving homeomorphisms of S up to isotopy. The mapping class group is finitely generated by the so called Dehn twists arround $2g+1$ well chosen curves. A Dehn twist arround any simple closed curve geometrically cuts the surface along the curve and glues back after making a 2π twist. This gives a well defined homeomorphism up to isotopy (which is all we need).



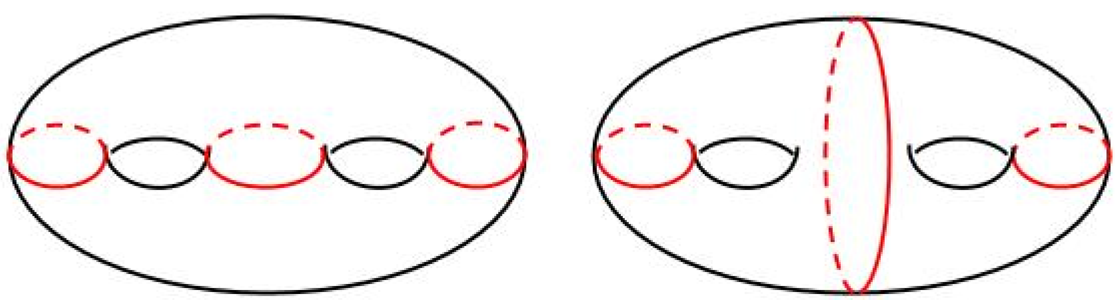
The mapping class group acts on Teichmuller space via $\varphi \cdot [X, f] = [X, f \circ \varphi^{-1}]$ and the quotient space is the moduli space of all hyperbolic metrics on S up to isometries. Thurston liked to imagine dressing the surface with some rigid all-in-one clothing assembled with zips, and then say that a point in moduli space corresponds to the clothing itself while Teichmuller space also records how it is worn. Unzipping and adding a twist moves the point in Teichmuller space but not the intrinsic measure of length in the surface, so the point in moduli space remains unchanged.

Length functions If α is a homotopy class of closed curves in S then $f_*(\alpha)$ is a well defined class of curves in X and corresponds, via the holonomy map, to a hyperbolic translation of $\mathbb{H}$. Actually, this translation is well defined up to conjugation (because $\mathscr{C}$ is identified with the set of conjugacy classes of the fundamental group), and the ambiguity amounts to choosing one of the lifts of $f_*(\alpha)$ as an axis for the hyperbolic translation. Note that the trace (translation parameter) $tr\,(f_*\alpha)$ is well defined on a conjugacy class of translations and the length of the geodesic for the corresponding metric $l_\alpha[X,f]$ is also well defined because equivalent points in Teichmuller space correspond to conjugate maps f_* in Γ. They are related by the formula:

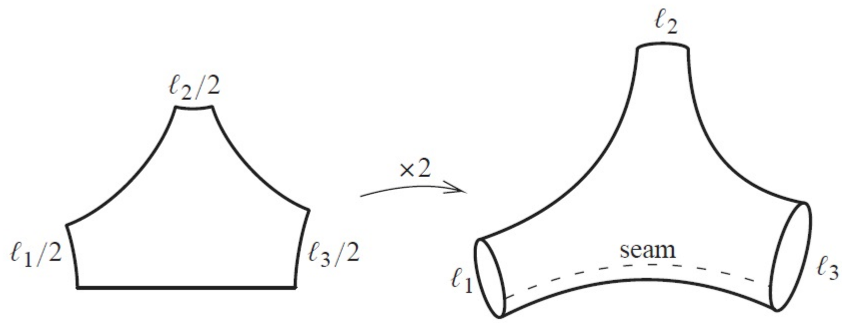
$$\left|\frac{tr\,(f_*\alpha)}{2}\right| = \cosh\left(\frac{l_\alpha[X,f]}{2}\right)$$

The maps $(l_\alpha)_{\alpha\in\mathscr{S}}$ give rise to $L\colon\ [X,f]\in\mathcal{T}(S)\longrightarrow(l_\alpha[X,f])_{\alpha\in\mathscr{S}}\in\mathbb{R}_+^{\mathscr{S}}$ which turns out to be a proper embedding, but the next paragraph proves an even more precise result.

Fenchel-Nielsen coordinates A closed surface S of genus g can be decompsed into $2g-2$ pairs of pants with geodesic boundary by choosing a maximal set of non intersecting $3g-3$ simple closed geodesics (α_i). A hyperbolic metric of the surface is uniquely determined by the metric on each pair of pants, and the gluing parameters encoding which congruent boundaries do we glue together and with what angles. But the set of decompositions into pairs of pants is discrete (lying inside $\mathscr{S}^{3g-3}$) and there is a unique hyperbolic structure on a pair of pants with geodesic boundary components of assigned length.



This last fact can bee seen by cutting the pair of pants along its seams (the geodesics minimising the distance between pairs of boundary components, which are unique by negative curvature) to obtain two right angled hyperbolic hexagons with three non incident sides of prescribed length; and such hyperbolic hexagons are unique up to isometry. This proves that a hyperbolic metric is uniquely determined by the lengths of $3g-3$ simple geodesics along with the glueing angles (mod 2π).



For a point in Teichmuller space the twisting parameters should not be mod out by 2π so that we get a set of coordinates in $\mathbb{R}^{3g-3} \times \mathbb{R}_+^{3g-3}$. The twisting parameters can be replaced by recording the lengths of dual simple curves (β_i) to the (α_i) such that $i(\alpha_i, \beta_j)$ is one or two if $i = j$ and zero otherwise, along with those of the Dehn twists $\left(\beta_i' = T_{\alpha_i}\beta_i\right)$. So Teichmuller space embedds in $\mathbb{R}_+^{9g-9} \subset \mathbb{R}^{\mathscr{S}}$.

Canoeing across geodesic currents

A simple question for lengthy thoughts

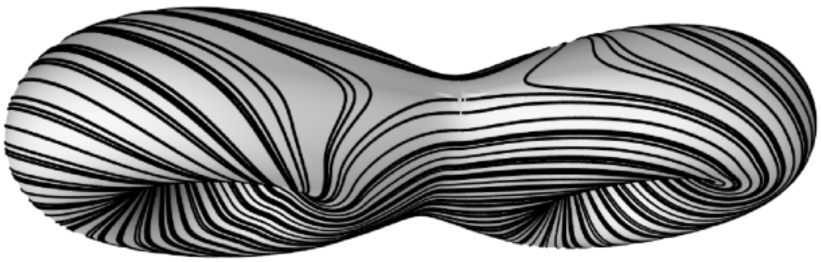
On a closed surface with negative euler characteristic $\chi = 2 - 2g$; metrics and closed curves share some common features. Indeed, negatively curved metrics can be evaluated on curves since there is a unique geodesic in each class. So there is some kind of duality operating here, and when there is a duality one wants to know its degree of separation.

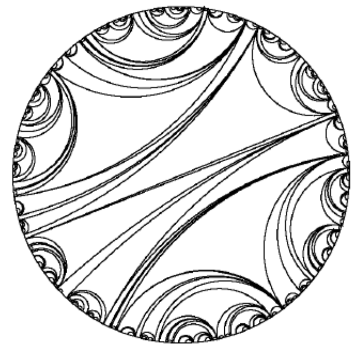
We have explained how geodesics and intersection numbers with those geodesics could enable one to compute geometric length in constant negative curvature. Better even, Fenchel-Nielsen coordinates show that a marked hyperbolic metric is uniquely determined by its evaluation on a set of $9g - 9$ simple curves. In [4], Otal proved that an isotopy class of a (not necessarily constant) negatively curved metric is uniquely determined by its marked length spectrum, that is the indexed family of all the closed curve's lengths. This result uses more simple curves to extract more information on the metric, and it is sharp in the sense that the indexing is vital to the statement's truth. So one can recover some informations about a metric by evaluating it on curves. What about the converse ? What information can one recover on a free homotopy class of curves by knowing its length for every hyperbolic metric ? Two curves are called length equivalent if they have the same length whatever the marked hyperbolic structure put on the surface.

One can also evaluate two curves $\gamma, \delta \in \mathscr{C}$ on one another by computing their intersection number $i(\gamma, \delta)$. And we have similar results: any $\delta \in \mathscr{C}$ is determined by $(i(\gamma, \delta))_{\gamma \in \mathscr{C}}$ and any $\delta \in \mathscr{S}$ is determined by its intersection numbers with $9g - 9$ well chosen curves. We also have similar questions: what does$(i(\gamma, \delta))_{\gamma \in \mathscr{S}}$ say for $\gamma \in \mathscr{C}$? Two curves are called simple intersection equivalent if those families are the same. This analogy between curves and metrics suggests they might fit in a common setting, and it is to describe that picture that that we introduce the following ideas. The relationships between length equivalence and simple intersection equivalence are the starting point of our common article with Moira Chas, where we give topological descriptions for those notions.

From simple curves to measured geodesic laminations

Measured laminations A geodesic lamination $\mathcal{L}$ of a Riemannian surface S is a a closed subset decomposed as a disjoint union of complete simple geodesics. Tipically, a geodesic lamination $\mathcal{L}$ contains a non countable set of non closed geodesics, each of them being dense in $\mathcal{L}$, along with a few isolated closed curves. A tranversal cut would then look like a Cantor set. Here's a graphical suggestion of a geodesic lamination in the genus two closed surface, along with its lift to the Hyperbolic plane.





A transverse measure λ on $\mathcal{L}$ is a famlilly of positive Radon measures $\left(\lambda_{|k}\right)_k$ defined on each arc $k : [0,1] \to S$ transverse to $\mathcal{L}$ which is compatible with restrictions to subarcs and invariant under holonomy : a homotopy between two transverse arcs k, k' maintaining the number of intersections with $\mathcal{L}$ constant all along, will identify λ_k with the pull back of $\lambda_{k'}$ by the overall transformation naturally induced by the homotopy. This holonomy property implies that the measures have support included in $\mathcal{L}$. We note $\mathcal{ML}(S)$ the set of measured geodesic laminations and $\mathbb{P}\mathcal{ML}(S)$ the corresponding projectivized set. We define on the space of measured laminations, the weak topology whose test functions are the continuous $f : k \to \mathbb{R}$ with compact support contained in a generic geodesic arc (ie transverse to all simple complete geodesics or equivalently, not contained in any simple complete geodesic) and the projectivized set is given the natural quotient topology.

Long simple curves Given a simple curve $\alpha \in \mathscr{S}$ and a positive number $t > 0$, one can define a measured lamination $\mu_{t.\alpha}$ by $(\mu_{t.\alpha})_{|k}(E) = card\,(E \cap \alpha)$ for every borelian set $E \subset k$ inside a transverse arc k to α. One could also consider weighted simple multicurves, that is positive linear combinations of simple curves which do not intersect each other, and their corresponding measured laminations $\sum t_\alpha \mu_\alpha$. It turns out that weighted simple multicurves, and even only weighted simple curves, are dense in the space of all measured geodesic laminations. One can construct very long simple curves by repeatedly applying Dehn twists to any initial simple curve. One can imagine that for some sequence of longer and longer simple curves uniformally filling the space (normalised by their geometric length), the associated sequence of measured laminations will eventually give the natural lebesgue measure on each transverse arc. This is very similar to the Crofton-flavoured formula's we presented.

A sphere of simple curves The intersection form $i(\cdot,\cdot)$ extends naturally to weighted simple curves by linearity and then to measured laminations by continuity. A first step towards our unification is identifying a measured lamination $\lambda \in \mathcal{ML}(S)$ with the dual function $i(\lambda,\cdot) \in \mathbb{R}_+^{\mathscr{S}}$. This identification map agrees with projectivization to give $\lambda \in \mathbb{P}\mathcal{ML}(S) \mapsto i(\lambda,\cdot) \in \mathbb{P}\left(\mathbb{R}_+^{\mathscr{S}}\right)$. If $\mathbb{R}_+^{\mathscr{S}}$ is endowed with the product topology and the projectivized set with the quotient topology, then this identification is actually an embedding and yields a homeomorphism from the set of projective measured laminations to a sphere $\mathbb{S}^{6g-7}$. This is one of the gems in the theory we owe to Nielsen-Thurston.

Geodesic currents

A geodesic current on S is $\pi_1(S)$invariant Radon measure on the set $G(\tilde{S})$ of all geodesics of the universal cover $\tilde{S} \cong \mathbb{H}$. Radon means Borelian (defined on the Borel's sigma algebra), locally finite (any point has a neighorhood of finite measure), and inner regular (the measure of a set is the supremum of the measures of the compact sets it contiains). The set of differences between two geodesic currents forms a vector space which we make into a topological space by endowing it with the weak uniform structure (defined by the family of semi-distances $d_f(\mu, \nu) = |\mu(f) - \nu(f)|$where f ranges in the compactly supported continuous functions on $G(\tilde{S})$). This defines our topology on the space of currents which we denote $\mathcal{C}urr(S)$.

As a first example, lets see how a measured lamination $(\mathcal{L}, \lambda)$ induces a geodesic current. All the lifts of $\mathcal{L}$ to $\tilde{S}$ form a $\pi_1(S)$ invariant subset $\tilde{\mathcal{L}} \in G(\tilde{S})$ and for any transverse arc $\tilde{k}$ to $\tilde{\mathcal{L}}$, we can define $\tilde{\lambda}\left(\{g \in G(\tilde{S}) | g \cap \tilde{k} \neq \emptyset\}\right) = \lambda_{|k}(k)$. This is enough to specify $\tilde{\lambda}$ as a Radon measure on $G(\tilde{S})$ and gives an embedding $\mathcal{ML}(S) \to \mathcal{C}urr(S)$ of the sphere of measured laminations into the space of currents.

Another source of examples is for $\gamma \in \mathscr{C}$, the set of its lifts $\tilde{\gamma}$ is a $\pi_1(S)$ invariant closed subset of $G(\tilde{S})$ and its counting measure (sum of dirac masses) defines a current μ_γ. We can similarly associate a geodesic current to a multicurve $\sum t_\gamma \gamma \in \mathbb{R}_+^{(\mathscr{C})}$ and those are dense in the space of all currents $\mathcal{C}urr(S)$. We can henceforth extend the intersection form $i(\cdot, \cdot)$ to $\mathcal{C}urr(S)$ and we can then prove that its light cone is the set of measured laminations: $i(\mu, \mu) = 0$ if and only if $\mu \in \mathcal{ML}(S)$.

Thurston's compactification Let's associate a geodesic current to a marked hyperbolic metric $[X, f]$. Such a metric lifts to the universal cover and then to its unit tangent bundle $T_1\tilde{S}$. The corresponding volume form can be contracted with the geodesic vector field to get a 2 form, giving a transverse measure to the geodesic flow which can be in turn interpreted as a mesure on $G(\tilde{S})$: the Liouville measure associated to the metric. Since $\pi_1(S)$ acts by isometries the measure is invariant. It is also borelian and inner regular since it arises from a 2 form.

The Liouville measure defines an embedding of $\mathcal{T}(S) \cong \mathbb{B}^{6g-6}$ into $\mathcal{C}urr(S)$ and it is compactified by the sphere of measured laminations. This compactification is equivariant with respect to the mapping class group's action. This means that for any mapping class $\varphi \in MCG(S)$, if a sequence of marked metrics $[X_n, f_n]$ converges to a projective measured lamination $\mu \in \mathbb{P}\mathcal{ML}(S)$ then $[X_n, f_n \circ \varphi^{-1}]$ converges to $\varphi \cdot \mu$ where this last action of a mapping class on a projective measured lamination coincides both with the continuous prolongation of its action on $\mathscr{S}$ (dense subset) and as the pull back of the measure μ once interpreted as a current.

The intersection form between a marked metric λ and a curve γ is the length of the curve for that metric. Moreover, Bonahon proved that a sequence of metrics λ_n converges to a projective measured lamination α if and only if for all curves $\gamma, \gamma' \in \mathscr{C}$,

$$\frac{\lambda_n(\gamma)}{\lambda_n(\gamma')} \longrightarrow \frac{i(\alpha, \gamma)}{i(\alpha, \gamma')}$$

Therefore the smooth function $\Phi_{\gamma,\gamma'}(\cdot) = \frac{i(\cdot,\gamma)}{i(\cdot,\gamma')}$ defined on $\mathcal{T}(S)$ can be continuously extended to $\partial\Phi_{\gamma,\gamma'}$ on $\mathbb{P}\mathcal{ML}(S)$. This function $\Phi_{\gamma,\gamma'}$ provides a better insight on the relationship between length equivalence and simple intersection equivalence. Indeed, they respectively amount to asking for $\Phi_{\gamma,\gamma'} \equiv 1$ and $\partial\Phi_{\gamma,\gamma'} \equiv 1$ and this is coherent with the fact that length equivalence implies simple intersection equivalence.

Conclusion

In [2] we therefore think of simple intersection equivalence as length equivalence at infinity, and make this more precise by specifying the algebraic behaviour of the intersection form at infinity: the intersection number between a curve γ and a simple curve α is akin to a tropical limit of the length of γ for a sequence of marked metrics converging to α. While the Fricke polynomial characterises length equivalence, we provide a tropical version of that polynomial characterising simple intersection equivalence. We derived this from our state sum formula drawing a parallel between the Fricke polynomial of length equivalent curves and the Jones polynomial of mutant knots.

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